THE

MATHEMATICAL GAZETTE.

EDITED BY

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ON THE TEACHING OF INDICES AND SURDS.

[*Read before the Mathematical Association, Saturday, January 27th,* 1900.]

The object of the present paper is to invite discussion as to the safest lines upon which to proceed in the teaching of Indices and Surds. And firstly, I am inclined to think that the beginner would get a firmer grasp of the theory of fractional indices (with which alone I am concerned) if a short course of surds, or, at least, of surd-forms, were made to precede; the simple theorem

$$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$$

would suffice, as shown later: and it seems more intelligible than its rival

$$a^{\frac{1}{n}}\times b^{\frac{1}{n}}=(ab)^{\frac{1}{n}}.$$

Next and independently, should we teach the theory of indices synthetically or analytically? In the first way fractional indices present themselves quite naturally after the first principles of Evolution have been acquired (see how quickly they are dealt with in the Algebra of the *Encyc. Brit.*); thus

$$\sqrt[5]{a^{15}}=a^{3=\frac{15}{5}},\quad \sqrt[5]{a^{20}}=a^{4=\frac{20}{5}}$$

suggests the definitions:

$$\text{let } a^{\frac{17}{5}} \text{ denote } \sqrt[5]{a^{17}},\ a^{\frac{p}{q}}, \text{ etc.}$$

It then only remains to prove that this extended definition of a^m will enable us to work with it in accordance with the old laws of indices proved for integers. This is sufficiently simple: but I have often observed what a snare it is for the unwary. Forty per cent. of those who rely on the ingenuity of the moment will somewhere in the course of the proof assume *id q.e.d.*

I digress, for a moment, to give the form of proof I have found most easily retained.

To prove $a^m \times a^n = a^{m+n}$, when $m = \frac{p}{q}$, $n = \frac{r}{s}$, etc.

Let $\sqrt[qs]{a} = x$ or $x^{qs} = a$,

then $a^m = \sqrt[q]{(x^{qs})^p} = \sqrt[q]{x^{qsp}} = x^{sp}$,

so $a^n = x^{qr}$,

$\therefore \quad a^m \times a^n = x^{sp+qr}$.

Again, $a^{m+n} = a^{\frac{ps+qr}{qs}}$

$= \sqrt[qs]{a^{ps+qr}}$

$= \sqrt[qs]{x^{qs(ps+qr)}}$

$= x^{ps+qr}$.

$\therefore$ etc.

This may be made neater if the student knows that

$$\sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

(But with the proposition in surd-forms to which I have alluded above, the exposition is still clearer. Thus

$$a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}} \times a^{\frac{qr}{qs}}$$

$$= \sqrt[qs]{a^{ps}} \times \sqrt[qs]{a^{qr}}$$

$$= \sqrt[qs]{a^{ps+qr}}$$

$$= a^{\frac{ps+qr}{qs}}$$

$$= a^{\frac{p}{q}+\frac{r}{s}}.)$$

The same substitution gives an easy proof of the second law

$$(a^m)^n = a^{mn}.$$

The plan seems to have the common-sense advantage of obtaining the values of all the quantities required.

Similarly, negative indices are *suggested* naturally in the usual way, then *defined*, and, as before, the index laws proved to be undisturbed. The analogy with the case of decimals in Arithmetic should be noted.

I next turn to the analytical treatment, in which the beginner is, I think, prematurely invited to participate in the joys of the investigator. I quote from the latest Algebra:

"The principle that we are guided by in extending our definition of index is an important one, known as the Principle of the Permanence of Equivalent Forms.

This principle consists in the assumption that a law of Algebra which admits of proof subject to certain limitations is true generally, provided that the removal of the limitations is not *incompatible* with the truth of the law.

Ex. 1. Find *the* meaning of $a^{\frac{1}{2}}$.

Since the formula $a^m \times a^n = a^{m+n}$ has been accepted as true for all values of m and n, $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a' = a$. Thus $a^{\frac{1}{2}}$ *is one* of the square roots of a.

Ex. 2. Find *a* meaning for a^{-1}, etc."

I resist the temptation to criticise, but I inquire, what end is served by this introduction to the hidden mysteries of science? Let us suppose the youthful investigator arrived at logarithms, and anxious to find out the "meaning" of $\log(-1)$. He reasons thus: By the P. of the P. of E. F.,

$$\log(-1) + \log(-1) = \log\{(-1)\times(-1)\} = \log 1 = 0;$$
$$\therefore \log(-1) = 0,$$

and hands his slate up.

Teacher. I am sorry to say that your conclusion is incorrect. I will explain when we come to Imaginaries.

The Y. I. But isn't *one* value zero?

Teacher (after reflection). No!

The Y. I. Then it is a case of incompatibility! (No answer.)

When they reach Imaginaries the delighted Y. I. has another shot, and brings, in triumph,

$$\begin{aligned} 2\log(-1) &= \log 1 \\ &= 2m\pi i, \text{ where } m, \text{ etc.}; \\ \therefore \log(-1) &= m\pi i. \end{aligned}$$

Teacher. Better, but m is restricted to be an odd integer!

The Y. I. Why?

Teacher. I will show you; only, give up that P. P. E. F.!

To return: in the analytical treatment the word "meaning" should be removed altogether, and "value" put in its stead. What meaning could there ever be for $a^{\sqrt{2}}$?

In another branch it is convenient to give $\underline{|0}$ the value 1; analytically, it has no meaning, though in that branch we might say that it was the number of permutations of no things taken none at a time!

We are at once confronted with the difficulty of many-valued functions. To say nothing of imaginaries, we have the two real values, in Algebra, of a square root. Should we face the difficulty and say, at this stage, that the index laws themselves must be extended? The most enthusiastic do not do that. Is not this abstention a reason for preferring the synthetic treatment? However, sooner or later the extension must come, viz.,

$$\text{a value of } a^m \times \text{a value of } a^n \text{ gives a value of } a^{m+n}.$$

I remember that when this form of statement first suggested itself to me I thought it something new, but only to find later that it had been given years ago by that gifted teacher De Morgan. I am glad that it is now to be found in several modern Algebras. It is evident that this extension should have its place between the elementary treatment and the study of such questions as "find all the values of $(-1)^{\frac{1}{3}}$."

It will now be seen that I am advocating the following order:

(1) Synthetic treatment of anomalous forms for a first course.

(2) Analytical treatment of ditto for a much later course, and only then in the case of students who are about to proceed to the study of imaginaries.

Some order should be adopted if only to secure fair play between examiner and examinee; it is hard on those who have followed the advanced course (including imaginaries) to be called upon to "prove that $a^{\frac{2}{3}}$ has only one admissible value," or, as in the B.A., London, 1895, to "define the meaning of the symbol a^n, where n is not necessarily a positive integer; interpret, in accordance with your definition, the symbols a^0, $a^{\frac{p}{q}}$, where p and q denote positive integers." I suppose the answer expected (?) was: The meaning (!) of a^m is that it is a symbolic something which obeys the index laws. But is not this skill in definition too much to demand from the passee? When shall we have "training of examiners"?

R. W. GENESE.

REVIEW.

Eléments de la Théorie des Nombres. Par E. CAHEN. Paris: Gauthier-Villars. 1900. Pp. viii., 404.

This book deserves a warm welcome from all arithmeticians, and especially from the amateurs for whom it is particularly intended. The analytical theory of numbers, in its higher branches, makes serious demands upon the student's mathematical resources: this is especially the case in those parts which have originated from, or been suggested by, transcendental analysis or function-theory, and have not yet been reduced to an essentially arithmetical form. But the elements of arithmetical science have been gradually brought, by the labours of many, to a very high degree of perfection; and this part of the subject, while eminently attractive to congenial spirits, has the advantage of requiring from its votaries no previous technical training whatever. The notions of integral, fractional, and irrational numbers; the laws of arithmetical operations; the elementary theory of residues and congruences; the elementary theory of binary quadratic forms: all these subjects are full of interest and beauty, and can be appreciated by anyone who has a sufficient stock of common-sense, and has not been hopelessly spoiled by a pernicious course of school arithmetic.

It is to such persons that Professor Cahen's book is principally addressed. By restricting himself to the range indicated above, he has been able to give clear and detailed explanations and provide a sufficient number of numerical examples. These, of course, should be supplemented by the reader's own exertions; there is no other subject, perhaps, in pure mathematics, which depends so much for its proper assimilation upon the construction, *by the student*, of illustrative examples. As remarked by H. J. S. Smith, the solution of an arithmetical problem consists, not in a formula, but in a rule of calculation; and the meaning of an algorithm is never fully appreciated until it has been actually applied to a series of particular cases.

The appearance of this work and of some others of a similar character (for instance, those of Tchébicheff, Wertheim, and Bachmann) seems to indicate a growing interest in the fascinating subject of arithmetic. It is to be hoped that in course of time the kind of arithmetic taught in schools will be altered in character. At present, it is too often of a thoroughly useless and even harmful description; rules are instilled and principles neglected (in default of principles, mere mechanical rules are misnamed principles: thus we have the "Principle of Cancelling"!); and much valuable time is wasted upon fantastic examples of no practical or educational value whatever. Any substantial improvement must be prepared for by some sort of understanding between teachers and examiners. So long as the prevailing types of questions are set, boys will be drilled in preparation for them; while, on the other hand, an examiner in arithmetic for a recognised examination cannot suddenly introduce an entirely different kind of paper, however desirable he may think it to be.

Mr. Workman's review in No. 20 of the *Gazette* has anticipated some remarks which I should otherwise have made about the absurd and artificial barrier set up between arithmetic and algebra. Algebra is, in the first instance, a generalized arithmetic; as soon as a boy really understands a *general* arithmetical truth, he can appreciate its shorthand statement in an algebraic formula, and this he should at once be taught—of course with due emphasis upon the fact that the letters stand for whole numbers, or rational numbers, or whatever the case may be. In this way a good deal of elementary algebra can be acquired without any extra effort worth mentioning; and it is the best introduction to the subsequent formal treatment of the subject. Such things as interest, proportion, and square root are really easier to understand when the symbolism of algebra is introduced. Again, the formulæ of mensuration and physics afford capital training preliminary to the systematic study of analysis, besides being of really practical importance and interest.

In conclusion, it may be remarked that Professor Cahen's book is distinguished by great lucidity and elegance, and that his hint of a more extensive treatise to follow gives rise to agreeable anticipations. Finally the tables of primes, indices, etc., at the end of the book (taken, by permission, from Tchébicheff) will be found of great value in making those numerical applications which, as already remarked, are indispensable if real progress is to be made. G. B. MATHEWS.

MATHEMATICAL NOTE.

80. *On the fundamental proposition connected with the vanishing of a Determinant.*

In proving the fundamental proposition that the vanishing of a determinant is the necessary and sufficient condition that the corresponding system of homogeneous linear equations may be satisfied by a set of values of the variables not all zero, it seems to be often thought sufficient to proceed somewhat as follows. The equations (n in number) being

$$a_1x_1+a_2x_2+\ldots=0,\quad b_1x_1+\ldots=0,\quad \text{etc.},$$

we infer by multiplication by $A_1, B_1, \ldots$ and adding, then by $A_2, B_2, \ldots$ and adding, and so on, that $\Delta x_1=0$, $\Delta x_2=0$, etc., so that $\Delta=0$ is *necessary* if any of the n quantities x_1, x_2, etc., is to be different from zero. *Conversely, if* $\Delta=0$ *the equations are satisfied by* $x_1=A_1$, $x_2=A_2$, etc.

Now it is true that if A_1, A_2, etc., should be all zero, we may take instead B_1, B_2, etc., or any other such series, but it is quite possible that all the first minors may be zero, and then, although the conclusion is really something of an *a fortiori* one, the above proof breaks down. The proposition being of fundamental importance, the following is suggested as a simple rigorous proof that, if a determinant vanishes, the corresponding system of equations must be satisfied by a set of values of the variables not all zero.

We will suppose generally that *all* determinants of the r^{th} order that can be formed from the *first* r columns of Δ vanish, but *not* all of the $\overline{r-1}^{\text{th}}$ order from the first $r-1$ columns. In particular, suppose the rows so arranged that the determinant consisting of the *first* $r-1$ *rows and columns* does not vanish, *i.e.* writing $l_1, l_2, \ldots l_r$ above the first r columns, as below, and denoting by $L_1, \ldots L_r$ the first minors with proper signs, *i.e.* the cofactors of $l_1, \ldots l_r$, in the determinant of the r^{th} order of which this is the first row, *we suppose* L_r *not zero.*

$$\begin{array}{l} \;\; l_1,\; l_2, \ldots l_r \\ \left| \begin{array}{l} a_1,\; a_2, \ldots a_r, \ldots a_n \\ b_1,\; b_2, \ldots b_r, \ldots b_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ k_1,\; k_2, \ldots \end{array} \right| . \end{array}$$

But $x_1=L_1$, $x_2=L_2, \ldots$, $x_r=L_r$, $x_{r+1}=0=x_{r+2}=\ldots=x_n$ will satisfy the n equations, the first $r-1$ of them identically, the remainder in virtue of the original supposition that all the determinants of the r^{th} order formed from the first r columns of Δ vanish. Thus the equations are satisfied by a system of values not all zero, for L_r is not zero by supposition.

The result therefore rigorously follows (1) if $\Delta=0$, but not all of A_1, B_1, $C_1, \ldots$, or (2) if all these vanish, but *not* all determinants of the $\overline{n-2}^{\text{th}}$ order, from the first $n-2$ columns, and so on. Thus it holds in all possible cases; for the extreme case to which we could be driven would be that where the vanishing of $a_1, b_1, \ldots k_1$ was in question. We may perhaps say that these cannot all vanish, or the given equations would not as supposed involve n variables; but even if we allow that we may be driven to this as a limiting case, the result still holds, for $x_1=$anything finite,

$$x_2=x_3=\ldots=0,$$

will satisfy the equations in this case. Thus the proof is complete.

For shortness we have kept our attention to the *first* r, $r-1$, etc., columns; but it is obvious that we need go no further if *any* determinant of the $\overline{r-1}^{\text{th}}$ order fails to vanish, which is a minor of the original determinant.

PERCY J. HEAWOOD.

PROBLEMS.

350. [K. 2. a.] The triangle formed by the Simson lines of A', B', C' with respect to a triangle ABC is similar to the triangle DEF, where A', B', C' are the mid points of BC, CA, AB respectively, and D, E, F the points of contact of the inscribed circle of ABC. (Corrected.) W. J. G.

355. [K. 20. e.] a, b, c are the centres of the squares described (externally) on the sides BC, CA, AB of the triangle ABC. D, E, F are the feet of the perpendiculars from A, B, C on these sides. H is the orthocentre of ABC. Tangents to the circles $HEAF$, $HFBD$, $HDCE$ are drawn from a, b, c; prove that the sum of the squares of these 18 tangents $=4\Delta(2\cot\omega+3)$. (Corrected.) R. Tucker.

360. [K. 10. e.] In a semicircle ABA', centre C, a chord $A'B$ is drawn, and a radius parallel to $A'B$. Find the locus of intersection of CB, $A'D$. C. Bickerdike.

361. [J. a. α.] At the corners and the middle points of the sides of a square, side 400 yds., are placed 8 guns. An officer, starting from the centre of the square, inspects the guns successively, but never two guns consecutively in the same side of the square. Find the least distance he has to walk. [$\sqrt{2}=\frac{7}{5}$; $\sqrt{5}=\frac{9}{4}$.] R. F. Davis.

362. [K. 10. e.] If an arc AB of a circle be divided equally in M and unequally in P, find a relation between the chords PA, PB, PM, AM analogous to Euc. II. 5. W. J. Johnston.

363. [L¹. 3. a.] Given two points on an ellipse and the positions of its axes, construct the lengths of the axes. A. Lodge.

364. [K. 5. a.] Points E, E' are taken in the triangles ABC, $A'B'C'$ respectively, such that $B\hat{E}C=B'\hat{E}'C'=A+A'$; $C\hat{E}A=C'\hat{E}'A'=B+B'$. Show that the lines EA, EB, EC; $E'A'$, $E'B'$, $E'C'$ divide the two triangles into mutually similar pairs of triangles. R. F. Muirhead.

365. [K. 20. d.] An ass is tied to a peg in the centre of a rectangular plot of grass, sides a, $2a$; if he can graze over just half the area, show that the length of his halter must be $\frac{1}{2}a$ cosec $59^\circ\, 4'\, 51{\cdot}6''$ very nearly. C. E. M'Vicker.

366. [K. 3. b.] ABC, DBC are two equilateral triangles on the same base BC. A point P is taken on the circle DBC. Show that PA, PB, PC are the sides of a right-angled triangle. E. M. Radford.

367. [J. 2. c.] A straight line of length unity is divided at random into three parts, subject to the condition that they may form the sides of a triangle. Show that the expectation of the area of the circle inscribed in the triangle is $\frac{\pi}{240}$. W. Allen Whitworth

368. [K. 20. e.] If P be any point in the plane of a triangle ABC, then shall $\Sigma a^2 . AP^4 - 2\Sigma BP^2 . CP^2 . \cos A + a^2b^2c^2=0$. (Deduce from 357.) A. C. L. Wilkinson.

369. [K. 5. d.] If triangles ABC, $A'B'C'$ be in perspective (centre P, axis DEF), and be also orthologic—*i.e.* such that perpendiculars from the vertices of either on corresponding sides of the other concur (in Q and Q')—then PQQ' is a straight line perpendicular to DEF.

Cor. 1. If PQ be points in the plane of a triangle ABC such that perpˢ from Q on AP, BP, CP, cut BC, CA, AB on a straight line, then so also do perpˢ from P on AQ, BQ, CQ; and both the lines are perp. to PQ.

Cor. 2. If $ABCD$ be four points determining a rectangular hyperbola, the tangent at D may be had thus:—Draw DE perp. to BD cutting CA at E, and DF perp. to CD cutting AB at F; then DT perp. to EF is the tangent. C. E. Youngman.

SOLUTIONS.

UNSOLVED QUESTIONS.—57, 129, 144, 152, 171, 175, 252, 271, 275, 279, 283, 285, 287, 306-8, 311, 312, 320, 326-7, 336-8, 341, 344-7, 349.

Solutions of these questions, and of 360-9, should reach the Editor before June 15th. They will be published as space is available.

The question need not be re-written; the number should precede the solution. Figures should be very carefully drawn to a small scale on a separate sheet.

111. [R. 4. c.] *Three uniform rods a, b, c, of the same thickness and density, are freely jointed at their extremities to form a triangle 1 2 3, which is supported at the middle point of the rod a. It is required to determine the position of equilibrium and the magnitude and direction of the reaction at the joint 1.* [General case of 111.]

Solution by W. J. DOBBS.

Let W_1, W_2, W_3 be the weights of the rods, proportional to their lengths, x the reaction of the hinge *1*, and a the angle between the rods b and c.

The resultant forces acting on the different rods, exclusive of the actions at the joints, are as indicated in the figure at the middle points of the rods.

The force diagram is shown on the right, the correspondence between the figures being indicated by means of the notation, which is that employed in Chap. XVIII. of my *Elementary Geometrical Statics.*

In the space diagram let the vertical through the hinge *3* meet the production of the rod c in H and the straight line drawn through the hinge *1* parallel to the rod a in K.

Then $$H1 : c = HK : K3 = CA : AB = W_2 : W_3 = b : c;$$
$$\therefore\ H1 = b;$$

$\therefore$ the rods b and c are equally inclined to the vertical.

This determines the position of equilibrium, and is the key to the construction of the force diagram, which can now be drawn to scale, giving a full graphical solution of the problem.

In the force diagram the triangles $BO1$ and COA are evidently equal in all respects.

Hence the line of reaction at the hinge *1* is, without being parallel to the rod a, equally inclined to the vertical.

Also, $$1\hat{O}B = A\hat{O}C = 3\hat{2}1;$$

$\therefore$ *the line of reaction at the hinge 1 is a tangent to the circum-circle of the triangle.*

Again, $$\frac{x}{\frac{W_2+W_3}{2}} = \frac{10}{BC} = \frac{AO}{BC} = \frac{K1}{H3} = \frac{\frac{ab}{b+c}}{2b\cos\frac{a}{2}}$$
$$= \frac{1}{2\cos\frac{a}{2}} \cdot \frac{a}{b+c}$$
$$= \frac{1}{2\cos\frac{a}{2}} \cdot \frac{W_1}{W_2+W_3};$$
$$\therefore\ x = \frac{1}{4} W_1 \sec\frac{a}{2}.$$

Hence if *1* is a right angle as in original form of question,
$$x = \frac{1}{2\sqrt{2}} W_1.$$

112. [R. 4. a.] *Two equal smooth spheres of radius r and weight W are placed inside a hollow cylinder, radius a, open at both ends, which rests on a horizontal plane; prove that, in order that the cylinder may not upset, its weight must be at least* $2W(1-r/a)$. (Clare and T. H., '96.)

Solution.

Let W' be the weight of the cylinder.

The reactions R, equating horizontal components, are equal.

Resolving horizontally for either sphere, $R=S\cos\theta.$ „ vertically, $W=S\sin\theta.$ $\therefore\ R=W\cot\theta.$

Taking moments about A,

$$R\,.\,r+W'\,.\,a=R(r+2r\sin\theta)\ \text{ and }\ \cos\theta=(a-r)/r\,;$$

$$\therefore\ W'a=2r\sin\theta\,.\,W\cot\theta=2rW\,.\,\frac{a-r}{r},$$

and
$$W'=2W\frac{a-r}{a}.$$

113. [R. 1. a.] *The coupling chain between an engine and a train, the mass of which is 96 tons, can bear a tension of 12 tons. Find the shortest time in which a speed of 30 miles an hour can be safely attained on a smooth level line.*

Solution by A. Lodge.

The impulse $Ft=$ momentum produced $=Mv$.

$$\therefore\ 12gt=96\times 30 \text{ miles per hour.}$$

$$\therefore\ t=\frac{8\times 30\text{ miles/hour}}{32\text{ ft./(sec.)}^2}$$

$$=\frac{8\times 30\times 5280}{32\times 60\times 60}\text{ seconds}$$

$$=11\text{ seconds.}$$

Or, maximum acceleration $=\dfrac{12\times 32}{96}=4$ ft. per sec. per sec.;

$\therefore$ if t be time required, $44=\dfrac{12\times 32}{96}t,$

and $t=11$ seconds.

114. [R. 9. b.] *A particle is projected from a fixed point above an inclined plane so to strike the plane at right angles; show that the square of the least possible velocity of projection is* $gc\{(\cos^2\alpha+4\sin^2\alpha)^{\frac{1}{2}}-\cos\alpha\}$, *where c is the perpendicular distance of the point from the plane, and α the inclination of the plane to the horizon.* (Gonv. and Caius, '95.)

Solution.

Let the particle strike the plane at a from its foot, starting at b from the foot, the join of starting point to foot being at β to the horizontal; and let $u\cos\theta$, $u\sin\theta$ be the horizontal and vertical velocities of projection.

Then we have, if t be the time of flight,

$$u\cos\theta\,.\,t=a\cos\alpha+b\cos\beta,$$

$$u\sin\theta\,.\,t=\tfrac{1}{2}gt^2+a\sin\alpha-b\sin\beta\,;$$

or eliminating a,

$$ut\sin(\alpha-\theta)+\tfrac{1}{2}gt^2\cos\alpha=2b\sin(\alpha+\beta)=2c. \quad\text{.............................(1)}$$

But $\dfrac{u\sin\theta - gt}{u\cos\theta} = -\cot\alpha$, because the particle strikes the plane at right angles.

Substituting for t in (1),

$$u^2\{[1+\cos(2\theta-2\alpha)]\cos\alpha - 2\sin(2\theta-2\alpha)\sin\alpha\} = 4gc\sin^2\alpha.$$

Then u^2 is a minimum when the large bracket is a maximum, *i.e.* when

$$\tan(2\theta-2\alpha) = -2\tan\alpha\ ;$$

$$\therefore\quad u^2\left\{\cos\alpha + \frac{\cos^2\alpha}{\sqrt{\cos^2\alpha+4\sin^2\alpha}} + \frac{4\sin^2\alpha}{\sqrt{\cos^2\alpha+4\sin^2\alpha}}\right\} = 4gc\sin^2\alpha,$$

or $$u^2\{\sqrt{\cos^2\alpha+4\sin^2\alpha}+\cos\alpha\} = 4gc\sin^2\alpha,$$

or $$u^2 = 4gc\sin^2\alpha\,(\sqrt{\cos^2\alpha+4\sin^2\alpha}-\cos\alpha)/4\sin^2\alpha$$

$$=\text{the given expression.}$$

115. [R. 9. b.] *Two equal smooth spheres of radius r move with the same speed in opposite directions in parallel lines at distance c apart; prove that the motion of each deviates on impact through a right angle if $c^2(1+e)=4er^2$.*

(Gonv. and Caius, '95.)

Solution.

With usual notation, $mv\cos\theta + mv'\cos\phi = mu\cos\alpha + m(-u\cos\alpha) = 0, \ldots(1)$

$$v\cos\theta - v'\cos\phi = -e[u\cos\alpha - (u\cos\alpha)] = -2eu\cos\alpha, \ldots\ldots\ldots\ldots(2)$$

$$v\sin\theta = u\sin\alpha, \ldots\ldots\ldots\ldots(3)$$

$$v'\sin\phi = u\sin\alpha, \ldots\ldots\ldots\ldots(4)$$

$$\theta = \frac{\pi}{2}+\alpha,\quad \phi = \frac{\pi}{2}-\alpha\ ;$$

$\therefore$ as (1) and (2) give $\quad v\cos\theta = -eu\cos\alpha = -v'\cos\phi$,

and (3) and (4) give $\quad \tan\theta = -\tan\phi = -\dfrac{1}{e}\tan\alpha$,

we get $\quad \alpha = \tan^{-1}\sqrt{e}\ ;$

$$\therefore\quad c^2 = 4r^2\sin^2\alpha = 4r^2e/(1+e).$$

116. [C. 1. g.] *If x, y be the rectangular coordinates of any point on any rectangular hyperbola having its centre at the origin*

$$(x^2+y^2)\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^3 - y\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0.$$

(Trin. (C.), '95.)

Solution.

$a(x^2-y^2)+2hxy+c=0$ is any such rectangular hyperbola.

Then $$a(x-yy_1)+h(xy_1+y)=0$$

and $$a(1-y_1{}^2-yy_2)+h(2y_1+xy_2)=0\ ;$$

equating these two values of h/a and multiplying out we get

$$(x^2+y^2)y_2 + (y_1{}^2+1)(xy_1-y)=0,$$

the required result.

121. [K. 20. c. a.] *Solve the equation*

(i.) $\sin(60^\circ - \theta) + \sec(30^\circ + \theta) = 2$;

(ii.) $\tan\theta + \tan 2\theta + \tan 3\theta = 0$. (Edinburgh.)

Solution.

(i.)
$$\sin(60^\circ - \theta) + \sec(30^\circ + \theta) = 2,$$
$$\sin(60^\circ - \theta)\cos(30^\circ + \theta) + 1 = 2\cos(30^\circ + \theta),$$
$$\sin(60^\circ - \theta)\sin(60^\circ - \theta) + 1 = 2\sin(60^\circ - \theta),$$
$$\sin^2(60^\circ - \theta) - 2\sin(60^\circ - \theta) + 1 = 0,$$
$$\sin(60^\circ - \theta) = 1,$$
$$\cos(30^\circ + \theta) = 1,$$
$$\frac{\pi}{6} + \theta = 2n\pi,$$
$$\theta = \frac{2n-6}{6}\pi = \frac{n-3}{3}\pi.$$

(ii.) Solve $\tan 3\theta = -(\tan 2\theta + \tan\theta) = -\tan 3\theta(1 - \tan 2\theta \tan\theta)$;

$$\therefore\ \tan 3\theta = 0 \text{ and } \theta = \frac{n\pi}{3},$$

or,
$$1 = -1 + \tan 2\theta \tan\theta = -1 + \frac{2t^2}{1-t^2}, \text{ where } t = \tan\theta,$$
$$\therefore\ 2t^2 = 1,$$
$$\theta = n\pi \pm \tan^{-1}\frac{1}{\sqrt{2}}.$$

122. [A. 1. b.] *Given* $xyz \neq 0$, *and* $yz + ay + bz = zx + az + bx = xy + ax + by = 0$, *and* $xyz = \Sigma yz$, *prove that* $a + b + 3 = 0$. (E.)

Solution.

Multiplying in turn by x, y, z.

$$xyz + axy + bxz = xyz + ayz + bxy = xyz + axz + byz = 0$$
$$\tfrac{1}{3}(3xyz + (a+b)\Sigma xy) = \tfrac{1}{3}(3xyz + (a+b)xyz) = 0;$$
$$\therefore\ \text{as } xyz \neq 0,\ a + b + 3 = 0.$$

127. [A. 3. k.] *Solve the equation*

$$(x^2 + bc)^2 = 4ax(x+b)(x+c).$$

Solution.

$$(x^2 + bc)^2 - 4ax(x^2 + bc) = 4ax^2(b+c),$$
$$(x^2 + bc - 2ax)^2 = 4ax^2(b+c+a),$$
$$x^2 - 2ax + bc = \pm 2x\sqrt{a(a+b+c)}, \text{ a quadratic.}$$

132. [K. 20. c. D. 2. β.] *Prove that*

(*a*) $\tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7} = 21$.

(*b*) $\tan^4\frac{\pi}{7} + \tan^4\frac{2\pi}{7} + \tan^4\frac{3\pi}{7} = 371$. (C.)

Solution.

(*a*) $\tan^2\frac{\pi}{7}$, $\tan^2\frac{2\pi}{7}$, $\tan^2\frac{3\pi}{7}$ are the roots of

$$z^3 - 21z^2 + 35z - 7 = 0. \quad (\textit{Loney}, \text{ Part II., p. 51.})$$

(*b*) L.H. $= 21^2 - 2 \cdot 35 = 371$.

153. [R. 1, a.] *A segment of a circle is placed with its base vertical, prove that the time from rest down any chord through the highest point is constant if the coefficient of friction has a certain fixed value.*

Note by A. Lodge.

The segment must be a minor segment.

If θ be the angle the chord AB makes with the vertical BC,

$$\tfrac{1}{2}mv^2 = mgs\cos\theta - mg\,.\,s\sin\theta\,.\,\mu.$$

Then if $\pi - BAC = a$, with the usual notation,

$$\tfrac{1}{2}m\dot{y}^2\frac{\sin(a-\theta)}{\sin a} = mg\,.\,y\,.\,(\cos\theta - \mu\sin\theta),$$

$$\therefore\ \tfrac{1}{2}m\dot{y}^2 = mgy \text{ if } \mu = \cot a.$$

177. [P. 3. b.] If A, B, C *are three circles, of which* A *is the inverse of* B *with respect to* C, *prove that the inverse of the circle of similitude of* A *and* B *with respect to* C, *is the radical axis of* A *and* B. A. Lodge.

Solution by Proposer.

Let O be the centre of inversion, *i.e.* the centre of C. Let OAB be a common tangent to A and B, cutting their circle of similitude in S, and their radical axis in R. We have to show $OS\,.\,OR = OA\,.\,OB$.

The range formed by the centres of the circles and the centres of similitude is harmonic,

$$\therefore\ \{OASB\} \text{ is harmonic.}$$

$$\therefore\ OS = \frac{2OA\,.\,OB}{OA+OB}.$$

R on the radical axis bisects AB,

$$\therefore\ OR = \tfrac{1}{2}(OA+OB), \qquad \therefore\ OR\,.\,OS = OA\,.\,OB.$$

182. [K. 1. b. B.] *Find the max. and min. values of the ratio of the sum of the medians of a triangle to the sum of the sides.*

Solution by G. Heppel.

Let the sides of the triangle be $1, p+x, p-x$.

If p be constant and x vary, there will be two triangles where the required ratio is the same (except when $p=0$), for we can interchange $p+x$ and $p-x$. Hence the position when $x=0$ is one of symmetry, and the ratio may be a max. or a min. Now the sides being $1, p, p$, we see that p may lie between $\frac{1}{2}$ and ∞, and no intermediate position gives a position of symmetry.

As p approaches $\frac{1}{2}$, the limiting ratio is $\frac{3}{4}$.

" ∞, " unity.

189. [K. 2. c.] *Find the area of the "nine point nonagon," when the triangle is* (i.) *acute,* (ii.) *obtuse.* W. J. Greenstreet.

Solution by Proposer.

(i.) Suppose $A > B > C$.

Let H; M_1, M_2, M_3; D, E, F; P, Q, R, be respectively the orthocentre, the vertices of the median and pedal triangles, and the mid pts. of AH, BH, CH.

The area of the nonagon is the area of $ABC - \Sigma(\triangle DBQ + \triangle QBM_3)$.

Hence area required

$$= \tfrac{1}{2}R^2\,\Sigma\sin 2A - \tfrac{1}{2}R\cos B\left(DB\cos C + \frac{c}{2}\cos A\right)$$

$$-\tfrac{1}{2}R\cos A(EA\cos C + AF\cos B) - \tfrac{1}{2}R\cos C\left(\frac{b}{2}\cos A + \frac{a}{2}\cos B\right),$$

which readily reduces to

$$\frac{R^2}{4}\{2\sin 2A + \sin 2B + 2\sin(A-C)\cos(A-B)\cos(B-C)\}.$$

(Queen's (C.), 1896.)

(ii.) (Lettering as before).

The area of the nonagon is that of the nonagon of the triangle BCH, and may be obtained directly as above, or more simply by writing

$$\pi - A,\ \frac{\pi}{2} - C,\ \frac{\pi}{2} - B,$$

for A, B, C respectively in result of (i.).

Thus area is $\frac{R^2}{4}\{\sin 2C - 2\sin 2A + 2\sin(A-C)\cos(A-B)\cos(B-C)\}$.

196. [A. 1. b.] *If a, b, c are three positive magnitudes, such that b lies between a and c, and $a \sim b > b \sim c$, and $b^{-1} \sim a^{-1} > c^{-1} \sim b^{-1}$, then the same inequalities hold with respect to A, B, C where $A = m + na^{-1}$; $B = m + nb^{-1}$; $C = m + nc^{-1}$, m and n being any positive magnitudes.* R. F. Muirhead.

Solution by Proposer.

$$(B \sim C)/(A \sim B) = (c^{-1} \sim b^{-1})/(b^{-1} \sim a^{-1}) < 1 \text{ by hyp.}$$

$$\therefore\ B \sim C < A \sim B.$$

Again, $\quad (C^{-1} \sim B^{-1})/(B^{-1} \sim A^{-1}) = A(B \sim C)/[C(A \sim B)]$.

$$= (m + ra^{-1})(c^{-1} \sim b^{-1})/[m + rb^{-1})(b^{-1} \sim a^{-1})] = (a + mr)(b \sim c)/[(bm + r)(a \sim b)].$$

The last formula shows that $C^{-1} \sim B^{-1} < B^{-1} \sim A^{-1}$ when $a \not> c$.

The last but one " " $<$ " " $a \not< c$.

265. [I. 2. b.] *If n be a composite number, what is its smallest value that satisfies the condition that $2^{n-1} - 1$ should be divisible by n?* G. Heppel.

Solution by Proposer.

Let $n = pq$, where p, q are primes.

$$N = 2^{pq-1} - 1 = (2^{q-1})^p - 1 + 2^{pq-p}(2^{p-1} - 1).$$

$$\therefore\ N \equiv M(q) \text{ if } 2^{p-1} - 1 \equiv M(q).$$

Similarly $\quad N \equiv M(p)$ if $2^{q-1} - 1 \equiv M(p)$.

Now $\quad 2^2 - 1 = 3$; $2^4 - 1 = 3 \cdot 5$; $2^6 - 1 = 3^2 \cdot 7$; $2^{10} - 1 = 3 \cdot 11 \cdot 31$

$$2^{20} - 1 \equiv M(2^{10} - 1) \equiv M(11), \quad \therefore\ n = 341.$$

Other values are $\quad 19 \times 73$; 17×257; 23×89.

266. [I. 2. b.] *In testing a number for possible divisors, it suffices to test the primes up to its square root.* E. Hill.

Solution by E. Fenwick.

If N is divisible by a prime $> \sqrt{N}$, the complementary factor must be $< \sqrt{N}$, and this would be discovered as stated.

278. [I. 2. b.] *The squares of two consecutive numbers are of the forms $11m + 4$, $12n + 1$; find the forms of the numbers.*

Solution by R. F. Davis.

If N be a number whose square is $(11m + 4)$, then $N = 11m \pm 2$.

If N' be a number whose square is $(12n + 1)$, then N' is $12n \pm 1$ or $12n \pm 5$.

Hence we have eight simple indeterminate equations:

$$11x + 2 = 12y \pm 1 \text{ or } 12y \pm 5.$$

$$11x - 2 = 12y \pm 1 \text{ or } 12y \pm 5.$$

The partial solutions, (24, 25), (42, 43), (46, 47), (64, 65), (90, 91), (108, 109), (112, 113), (130, 131), suggest the general forms

$$(132t + 24,\ 132t + 25).$$

Aliter:

$$12n+1 \text{ is odd, } \therefore\ 11m+4 \text{ is square of even number,}$$

$$\therefore\ m=M(4)\,;\ \sqrt{11m+4}=22p\pm 2.$$

Upper sign gives $121p^2+66p+2=3n$, whence $p^2-1=M(3)$, and $p=3r\pm 1$.

Lower sign gives $121p^2-11p \quad =3n$, whence $p=3r$ or $3r-1$;

$$\therefore \text{ forms are } 66r+24,\ 66r+25\ ;\ 66r-20, 66r-19\ ;\ 66r-2,\ 66r-1\ ;$$
$$66r-24,\ 66r-23.$$

291. [J. 2. f.] *There are* 10 *tickets:* 5 *blanks, and the others marked* 1, 2, 3, 4, 5. *What is the probability of drawing* 10 *in* 3 *trials* (*a*) *tickets replaced;* (*b*) *tickets not replaced.* (Actuaries, Ed., 1891.) R. F. MUIRHEAD.

Solution by Proposer.

(*a*) Tickets replaced. The possible drawings are 5, 5, 0; 5, 4, 1; 5, 3, 2; 4, 4, 2; 4, 3, 3, and their permutations, the number of these being 15, 6, 6, 3, 3, for there are 5 blanks. $\therefore$ chance is 33/1000.

(*b*) Tickets not replaced. The possible drawings are 5, 4, 1; 5, 3, 2; and we need not consider order. Hence chance is $2/{}_{10}C_3=1/60$.

298. [K. 20. c. a.] *Find the sum and product of the non-zero roots of*

$$\Sigma\, tan^{-1}(a-x)=\Sigma\, tan^{-1}a=tan^{-1}s. \qquad \text{(C.)}$$

Solution by J. F. HUDSON.

Since $\quad \tan \Sigma A\,(1-\Sigma \tan A \tan B)=\Sigma \tan A-\Pi \tan A,$

we have $\quad [\Sigma(a-x)-\Pi(a-x)]/[1-\Sigma \tan(a-x)\tan(b-x)]$

$$=(\Sigma a-abc)/(1-\Sigma ab)=s=\frac{\Sigma a-abc}{1-\Sigma ab}.$$

This reduces to

$$x^2(\Sigma ab-1)-x\Sigma(ab+1)(a+b)+\Sigma(a^2+1)(b^2+1)=0,$$

giving the sum and product required.

[The solution p. 295 assumed that $2s=a+b+c$.]

299. [J. 2. f.] *A bag contains a red, b yellow, and c blue balls. A ball is drawn and shown to X and Y. X, who speaks the truth once in x times, says it is blue. Y, who speaks the truth once in y times, says it is red. What is the consequent chance of its being yellow?* (Trinity (C.), 1897.)

To this Mr. Muirhead adds: *If A says that B says it is red, what is the chance it is yellow?*

Solution by W. ALLEN WHITWORTH.

I assume that the statements of X and Y depend upon their judgment of a ball submitted to them, and not on any guess. If guessing, the result would depend on whether they know the proportion of balls in the bag. I assume that they know all balls to be one of the three stated colours, and that their judgment must be either that it is red, or yellow, or blue. They cannot invent another colour. If, therefore, a red ball is submitted to X, the chance he calls it red is $\frac{1}{x}$; that he calls it not-red is $\frac{1-x}{x}$; that he calls it yellow is $\frac{1-x}{2x}$.

Similarly if he is reporting what Y said.

The ball is either red, yellow, or blue.

(1) If red, events have happened, the chance of which was $\dfrac{a}{\Sigma a}\cdot\dfrac{x-1}{2x}\cdot\dfrac{1}{y}$.

If yellow, „ „ „ „ $\dfrac{b}{\Sigma a}\cdot\dfrac{x-1}{2x}\cdot\dfrac{y-1}{2y}$.

If blue, „ „ „ „ $\dfrac{c}{\Sigma a}\cdot\dfrac{1}{x}\cdot\dfrac{y-1}{2y}$.

These are as $2a(x-1):b(x-1)(y-1):2c(y-1)$;

$\therefore$ chance of yellow ball is

$$b(x-1)(y-1)/[2a(x-1)+b(x-1)(y-1)+2c(y-1)].$$

(2) If red, events have happened, the chance of which is

$$\frac{a}{\Sigma a}\left[\frac{1}{y}\cdot\frac{x-1}{2x}+\frac{y-1}{2y}\cdot\frac{x-1}{2x}+\frac{y-1}{2y}\cdot\frac{1}{x}\right].$$

If yellow, „ „ „ „

$$\frac{b}{\Sigma a}\left[\frac{1}{y}\cdot\frac{x-1}{2x}+\frac{y-1}{2y}\cdot\frac{x-1}{2x}+\frac{y-1}{2y}\cdot\frac{1}{x}\right].$$

If blue, „ „ „ „

$$\frac{c}{\Sigma a}\left[\frac{1}{y}\cdot\frac{1}{x}+\frac{y-1}{2y}\cdot\frac{x-1}{2x}+\frac{y-1}{2y}\cdot\frac{x-1}{2x}\right].$$

Which are as $a(xy+x+y-3):b(xy+x+y-3):2c(xy-x-y+3)$;

$\therefore$ chance of yellow ball is

$$b(xy+x+y-3)/[(a+b)(xy+x+y-3)+2c(xy-x-y+3)].$$

310. [I.] *A and B are exploring a desert with the object of penetrating as far into the interior as possible. Each man carries provisions for* 20 *days' journey, find the greatest distance penetrated.* C. PENDLEBURY.

Solution by PROPOSER.

The question assumes that neither can carry more than 20 days' provisions. After 5 days' journey, each has 15 days' provisions left.

Then, B out of his stock gives 5 days' provisions to A, buries 5 days' provisions in a *cache*, and has just enough left to carry him home.

A with his new stock of 20 days' provisions can go alone 10 days' journey further. On his return to the *cache* this stock is just exhausted, and the supply in the *cache* will just be sufficient to carry him home.

The distance A penetrates is thus 15 days' journey.

317. [D. 6. b.] If $\dfrac{e^{ax}-1}{e^{a}-1}=x+aP_2(x)+\dfrac{a^2}{\underline{|2}}P_3(x)+\dfrac{a^3}{\underline{|3}}P_4(x)+\dots$

then $$-P_n(-x)=P_n(x)-(n-1)xP_{n-1}(x)+\frac{(n-1)(n-2)}{\underline{|2}}x^2P_{n-2}(x)+\dots$$
$$+(-1)^{n-2}(n-1)x^{n-2}P_2(x)+(-1)^{n-1}x^n.$$

(Trinity, 1896.)

Solution by C. E. M'VICKER; J. F. HUDSON.

If $\dfrac{e^{ax}-1}{e^{a}-1}\equiv f(x)$, then $f(-x)=-e^{-ax}f(x)$. ....(1)

But $\quad -f(-x)=x-a_2 \,.\, P_2(-x)-\dfrac{a^2}{\lfloor 2} \,.\, P_3(-x)-\ldots-\dfrac{a^{n-1}}{\lfloor n-1}P_n(-x)\ldots$

$$e^{-ax}=1-\frac{ax}{\lfloor 1}+\frac{a^2x^2}{\lfloor 2}-\ldots+(-1)^n\frac{a^nx^n}{\lfloor n}\ldots$$

$$f(x)=x+aP_2(x)+\frac{a^2}{\lfloor 2}P_3(x)+\ldots+\frac{a^{n-1}}{\lfloor n-1}P_n(x)\ldots$$

comparing coefficients of a^{n-1} in (1), and dividing by $\lfloor n-1$ on both sides we get the required result.

319. (*a*) [R. 7. b.] *A shot of mass m is fired from a smooth bore gun whose carriage moves freely on a level plain; the inclination of the gun is fixed at a, the impulse exerted by the powder is I, and the mass of gun and carriage is M. Find the range on the plane, and determine the initial velocity of the shot.*

(Trinity (C.), 1897.)

Solution by R. F. Muirhead.

Supposing the impulse to be exclusive of that of the barrel on the shot, then resolving momenta along the (smooth) barrel,

$$I=mu\cos(\theta-a), \qquad (1)$$

where θ is the inclination of path of shot to horizon, and u the initial velocity.

Next we have the kinematic relation

$$u/\sin a = V\sin(\theta-a), \qquad (2)$$

where V is the velocity of the gun. Finally, resolving momenta horizontally for gun and shot, we have $MV=mu\cos\theta$. Eliminating θ and V, we have

$$u^2=I^2(M^2+2Mm\sin^2 a+m^2\sin^2 a)/[m^2(M+m\sin^2 a)^2],$$

$$\text{range}=(M(M+m)I^2\sin 2a)/[gm^2(M+m\sin^2 a)^2].$$

321. [J. 1. c.] *If H_r denote the sum of the homogeneous products of r dimensions of n quantities $a, \beta, \gamma, \ldots$ then any function of their differences satisfies*

$$n\frac{du}{dH_1}+(n+1)H_1\frac{du}{dH_2}+\ldots=0.$$

E. P. Barrett.

Solution by C. E. M'Vicker.

If $a, b, c, \ldots$ are integers (zeros not excluded), and $\Sigma a=(r-1)$, then H_r may take any of the forms $\Sigma a^{a+1}\beta^b\gamma^c$, $\Sigma a^a\beta^{b+1}\gamma^c$, $\Sigma a^a\beta^b\gamma^{c+1}, \ldots$.

Hence $\Sigma\dfrac{dH_r}{da}=\delta\,.\,H_r$ (say) $=\Sigma(a+1+b+1+\ldots)a^a\beta^b\gamma^c\ldots=(n+r-1)H_{r-1}$.

Now $$\frac{du}{da}=\frac{du}{dH_1}\cdot\frac{dH_1}{da}+\frac{du}{dH_2}\cdot\frac{dH_2}{da}+\ldots;$$

$$\therefore\ \Sigma\frac{du}{da}=\delta H_1\cdot\frac{du}{dH_1}+\delta H_2\cdot\frac{du}{dH_2}+\ldots=n\frac{du}{dH_1}+(n+1)H_1\frac{du}{dH_2}+\ldots.$$

Hence if u be a function of the differences of a, β, γ,

$$n\frac{du}{dH_1}+(n+1)\frac{du}{dH_2}+\ldots+(n+r-1)H_{r-1}\frac{du}{dH_r}+\ldots+(2n-1)H_{n-1}\frac{du}{dH_n}=0.$$

Solution by J. F. Hudson.

If $a, \beta, \gamma, \ldots$ are roots of $x^n-p_1x^{n-1}+\ldots=0$.

Then $\quad H_1-p_1=0\,;\ H_2-H_1p_1+p_2=0\,;\ H_3-p_1H_2+p_2H_1-p_3=0\ldots;$

$$\therefore\ p_1=H_1\,;\ p_2=H_1^2-H_2\,;\ p_3=H_1^3-2H_1H_2+H_3\,;\ldots$$

and $\quad H_1=p_1\,;\ H_2=p_1^2-p_2\,;\ldots.$

Any function of the differences of $\alpha, \beta, \gamma, \ldots$ in terms of $p_1, p_2, \ldots$ satisfies

$$n\frac{du}{dp_1}+(n-1)p_1\cdot\frac{du}{dp_2}+(n-2)p_2\cdot\frac{du}{dp_3}+\ldots=0.$$

But $$\frac{du}{dp_1}=\Sigma\frac{du}{dH_1}\cdot\frac{dH_1}{dp_1}=\frac{du}{dH_1}+2p_1\frac{du}{dH_2}+(3p_1^2-2p_2)\frac{du}{dH_3}+\ldots,$$

so $$\frac{du}{dp_2}=-\frac{du}{dH_2}-2p_1\frac{du}{dH_3}-\ldots; \text{ and } \frac{du}{dp_3}=\frac{du}{dH_3}+\ldots.$$

Then

$$n\frac{du}{dp_1}+(n-1)p_1\cdot\frac{du}{dp_2}+\ldots=n\frac{du}{dH_1}+(n+1)p_1\frac{du}{dH_2}+(n+2)(p_1^2-p_2)\frac{du}{dH_3}+\ldots$$

$$=n\frac{du}{dH_1}+(n+1)H_1\frac{du}{dH_2}+(n+2)H_2\cdot\frac{du}{dH_3}+\ldots.$$

322. [L[2]. 2. d.] *A plane cuts a cone, vertex O, in an ellipse, focus S. The tangent to the ellipse at P meets the directrix in Q. Show that Q is the pole with respect to the focal sphere of the plane OSP.* J. A. BRERETON.

Solution by J. BLAIKIE.

Join OP and let it touch focal sphere in K. Let the axis of the ellipse meet the directrix in X. Then the polar plane of O is the plane KQX; the polar plane of K is the plane OKQ; and the polar plane of S is the plane SXQ. These three planes pass through Q, therefore the polar plane of Q is the plane OKS or OSP.

323. [K. 4.] *Construct an equilateral triangle, given the centres of its in-squares.* W. S. COONEY.

Solution by PROPOSER.

Let A', B', C' be the in-centres; $AA'P$ the perpendicular to BC; $A'R$ perpendicular to CA.

Let $AP=p$; $A'P=s=$half side of in-square; let $AA'P$ cut $B'C'$ in Q; then

$$B'T=A'R=\tfrac{1}{2}AA'=\tfrac{1}{2}(p-s),$$

$$AQ=p-B'T=\tfrac{1}{2}(p+s),$$

$$AQ/AA'=\tfrac{1}{2}(p+s)/(p-s)=\sqrt{3}/2; \quad \because \ s=p(2-\sqrt{3});$$

$$\therefore \ A \text{ is found from } \triangle A'B'C'; \quad \therefore \text{ etc.}$$

Solution by J. BLAIKIE.

Let ABC be the triangle, O, P, Q the given centres, G the centroid of ABC and OPQ. Let $AB=a$ and side of inscribed square $=b$. Then

$$a=b\left(1+\frac{2}{\sqrt{3}}\right); \quad \therefore \ b=(2\sqrt{3}-3)a, \quad ON=\tfrac{1}{2}b=(\sqrt{3}-\tfrac{3}{2}a), \quad AN=\frac{\sqrt{3}}{2}a,$$

$$GN=\frac{\sqrt{3}}{6}a, \quad AG=\frac{\sqrt{3}}{3}a, \quad OG=\frac{9-5\sqrt{3}}{6}a, \quad \frac{AG}{OG}=\frac{\sqrt{3}}{9-5\sqrt{3}}=3\sqrt{3}+5.$$

OG must therefore be produced till the part produced is $(3\sqrt{3}+5)OG$ in order to find A. Similarly for B and C.

Solution by C. E. YOUNGMAN.

The centres $A'B'C'$ are by symmetry the corners of an equilateral triangle; draw the altitude $A'D'$ and on it make backwards $A'D=A'B'+B'D'+D'A'$. Obtain similarly E and F from B' and C'; then DEF are the mid points of the sides of the required triangle ABC.

For, writing d, a, p, a', p' for $A'D$, BC, AD, etc., we have made

$$d=p'+\frac{3a'}{2}=p'(1+\sqrt{3}),$$

and the figure gives $\qquad d+\frac{2}{3}p'=\frac{1}{3}p.$

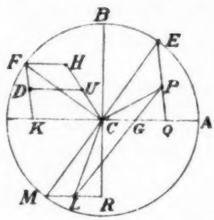
Hence, eliminating p', $\quad d=p/(2+\sqrt{3})=ap/(2a+2p)$;

$$\therefore\ DA'=\text{half side of in-square of } ABC.$$

324. [L[1]. 3. c.] *CP, CD are conjugate radii of an ellipse; CP is multiplied by $\frac{b}{a}$, and both are turned through $+90°$ into the lines CU, CL respectively. If CP be in the quadrant ACB, show that DU is parallel to CA and $PL=a+b$.* (Corrected.) R. W. GENESE.

Solution by W. S. COONEY; R. B. WORTHINGTON.

Let the ordinates PQ, DK cut the auxiliary circle in E, F.



$$PQ/EQ=DK/FK=b/a \text{ and } E\hat{C}F=90°.$$

If CP be turned through 90° to the position CH, and FH be parallel to CA, then from the triangles ECP, FCH we have $CH=CP$. If

$$CU=\frac{b}{a}.CH,\quad CU/CH=DF/FK=b/a,$$

$$\therefore\ DU \text{ is parallel to } CA.$$

Hence, *on CP only*, CU is taken $=\frac{b}{a}.CP$.

Draw CL perpendicular to CD; let EC cut the auxiliary circle in M; draw MLR parallel to CA, cutting CL perpendicular to CD in L and BC produced in R.

The figures $FCKD$, $MCRL$ are congruent; $\therefore\ CL=CD$.

Also $MR/RL=FK/DK=EQ/PQ$; $\therefore\ PL$ and EM are parallel.

But $LG=CM=a$; and $PG=b$; $\therefore\ PL=a+b$.

[The correction of (324) is necessary. If θ be eccentric angle of P, we have in vectors

$$CP=a\cos\theta+b\iota\sin\theta;\ CD=-a\sin\theta+b\iota\cos\theta;$$

$$CP-\iota CD=(a+b)\cos\theta+(a+b)\iota\cos\theta,$$

a vector of length $a+b$. Also

$$\frac{b\iota}{a}CP-CD=\left(-\frac{b^2}{a}+a\right)\sin\theta,$$

a real quantity, and therefore parallel to CA. (R. W. G.)]

325. [L[1]. 7. d.] (*a*) *Given a point on a rectangular hyperbola and a focus, show that the second focus lies on a Limaçon. If two points are given on a conic, and one focus, the second focus lies on an ellipse.*

(*b*) *Given that a conic of constant minor axis is inscribed in a triangle, the foci lie on a cubic.* W. J. GREENSTREET.

Solution by J. F. HUDSON; C. E. YOUNGMAN.

(*a*) Given (of any conic) S, P, and e, required the locus of S'. On PS make $SQ=SP$, and draw the circle PQS' cutting SS' at R. Then the triangles SPS', SRQ are similar; and $SP\pm S'P=SS'/e$;

$$\therefore\ SR\pm RQ=SQ/e;$$

$\therefore\ R$ describes a conic with foci S and Q and given eccentricity.

But $SR \,.\, SS' = -SP^2$;

$\therefore$ S' describes the focal inverse of a conic; *i.e.* a limaçon.

Given S, P_1, and P_2, required the locus of S'. Since

$$SP_1 \pm S'P_1 = SP_2 \pm S'P_2,$$

the locus is a hyperbola or ellipse according as P_1, P_2 are on the same branch or not of the varying conic.

(*b*) If α, α'; β, β'; γ, γ'; be the distances of S, S' from the sides of the triangle, it is given that $\alpha\alpha' = \beta\beta' = \gamma\gamma' = c^2$. But each of these

$$= 2\Delta \Big/ \left(\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}\right); \quad \therefore \ \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = \frac{2\Delta}{c^2} = \frac{4\Delta^2}{c^2(a\alpha + b\beta + c\gamma)};$$

$\therefore$ the trilinear equation of the locus is $\Sigma a\alpha \,.\, \Sigma a\beta\gamma = k^2 \alpha\beta\gamma$, which represents a cubic touching the circum-circle at the corners of the triangle.

Solution (*a*) by J. F. Hudson and Proposer.

Equation of conic referred to focus and major axis is

$$a(1-e^2) = r(1 + e\cos\theta) \text{ or } 2ae = 2er(1 + e\cos\theta)/(1-e^2).$$

If R be rad. vect. to second focus, $R = 2er/(1-e^2) + 2e^2r/(1-e^2)\cos\theta$;

$$\therefore \text{ locus is } R = A + B\cos\theta, \text{ a limaçon.}$$

[*Mr. R. F. Davis*, with usual notation, produces SP to Q making $SQ = 4SP$, and finds the locus to be the pedal with respect to S of a circle, centre Q, and constant radius. *M. Barisien*, for the ellipse, rectangular axes, and SA as axis of x, finds $r = 2a(\cos\theta \pm \sqrt{2})$, and notes that the directrices envelope the circles $(x-3a)^2 + y^2 = 9a^2/2$; $(x-a)^2 + y^2 = a^2/2$].

328. [K. 20. e.] *In a triangle ABC prove that*

$$\Sigma \sin 2A\,[R\cos(B-C) - 2r]^2 = 2\,(\Pi \sin A)\,(R-2r)^2.$$

H. M. Leslie.

Solution by C. E. Youngman.

Draw any transversal t, making angles α, β, γ with the sides of the triangle, and on it project A, B, C into D, E, F. Count α obtuse if D lies between E and F. Then $DE + EF + FD = 0$ gives

$$\Sigma a \cos\alpha = 0 \text{ or } \Sigma \sin A \cos\alpha = 0, \text{ ....(1)}$$

Draw the images of t for BC, CA, AB, forming a new triangle $A'B'C'$ with whose sides t makes the angles $2\alpha, 2\beta, 2\gamma$. The angles A', B', C' are $180° - 2A$, etc.; hence by (1) $\Sigma \sin 2A \cos 2\alpha = 0$. This gives $2\Sigma \sin 2A \sin^2\alpha = \Sigma \sin 2A$, and therefore

$$\Sigma \sin 2A \sin^2\alpha = 2\Pi \sin A. \text{ ....(2)}$$

Now if t pass through the in- and nine-point centres,

$$\sin\alpha = [\tfrac{1}{2}R\cos(B-C) - r]/(\tfrac{1}{2}R - r), \text{ and so on.}$$

Substitute in (2), and the given relation is proved.

329. [D. 2. b. β.] *If*

$$C_n = \frac{\cos n\Sigma\alpha}{\Pi \sin(\beta+\gamma)} + \Sigma \frac{\cos n(\beta+\gamma-\alpha)}{\sin(\beta+\gamma)\sin(\beta-\alpha)\sin(\gamma-\alpha)},$$

then $\quad c_0 = c_1 = c_2 = 0$; $c_3 = -32\Pi \sin\alpha$; $c_4 = -32\Pi \sin 2\alpha$.

F. S. Macaulay.

Solution by R. F. Davis.

Let $$\frac{-x^2}{\left(1-\frac{x}{abc}\right)\left(1-\frac{bcx}{a}\right)\left(1-\frac{cax}{b}\right)\left(1-\frac{abx}{c}\right)}=\frac{P}{1-\frac{x}{abc}}+\frac{Q_1}{1-\frac{bcx}{a}}+\dots.$$

Put $x=abc$; we find $P=1\Big/\Pi\left(bc-\frac{1}{bc}\right)$;

$x=a/(bc)$; we find $Q_1=1\Big/\left[\left(bc-\frac{1}{bc}\right)\left(\frac{b}{a}-\frac{a}{b}\right)\left(\frac{c}{a}-\frac{a}{c}\right)\right]$, etc.

Expanding in ascending powers of x:

$$P\left(\frac{1}{abc}\right)^n+Q_1\left(\frac{bc}{a}\right)^n+Q_2\left(\frac{ca}{b}\right)^n+Q_3\left(\frac{ab}{c}\right)^n=\text{coefficient of } x^n$$

in $$\frac{-x^2}{1-x\left(\frac{1}{abc}+\Sigma\frac{bc}{a}\right)+x^2\left(\Sigma a^2+\frac{1}{\Sigma a^2}\right)\dots+x^4}$$

$$=-x^2\left\{1+x\left(\frac{1}{abc}+\Sigma\frac{bc}{a}\right)-x^2\left(\Sigma\frac{1}{a^2}+\Sigma a^2\right)+\dots+x^2\left(\frac{1}{abc}+\Sigma\frac{bc}{a}\right)^2+\dots\right\}.\dots(1)$$

Now put $a=\cos\alpha+\iota\sin\alpha$, etc.,

$$P=1/[8\iota^3\Pi\sin\overline{\beta+\gamma}];\quad Q_1=1/[8\iota^3\sin(\beta+\gamma)\sin(\beta-\alpha)\sin(\gamma-\alpha)], \text{ etc.}$$

$$(1/abc)^n=\cos n\Sigma\alpha-\iota\sin n\Sigma\alpha,$$

$$\therefore\ c_n/[8\iota^3]=\text{imaginary part of coefficient of } x^n \text{ in (1)},$$

and $$c_0=c_1=c_2=0.$$

$$\frac{c^3}{8\iota^3}=-\iota[-\sin\overline{\alpha+\beta+\gamma}+\Sigma\sin(\beta+\gamma-\alpha)]=-\iota\,.\,4\Pi\sin\alpha,$$

$$\therefore\ c_3=-32\Pi\sin\alpha.$$

Similarly $c_4=-32\Pi\sin 2\alpha$, $\therefore$ etc.

330. [I. 2. b.] *If n be an integer prime to 2 and 3, n^2-1 is divisible by 24. If n be an integer prime to 2, 3, and 5, $(n^2-1)^2(n^2-4)$ is divisible by 8640.*

P. A. MacMahon.

Solution by J. Blaikie, and others.

Since n is prime to 2, $n-1$ and $n+1$ are both divisible by 2, and one of them is divisible by 4; $\therefore$ n^2-1 is divisible by 8. Also one of the 3 consecutive numbers $n-1$, n, $n+1$ is divisible by 3, and since it is not n $\therefore$ n^2-1 is divisible by 24. Also one of the consecutive odd numbers $n-2$, n, $n+2$ must be divisible by 3; $\therefore$ n^2-4 is divisible by 3, and one of the 5 consecutive numbers $n-2$, $n-1$, n, $n+1$, $n+2$ is divisible by 5;

$$\therefore\ (n^2-1)(n^2-4) \text{ is divisible by } 5;$$

$$\therefore\ (n^2-1)^2(n^2-4) \text{ is divisible by } 24^2\times3\times5=8640.$$

331. [L[1]. 5. a.] *The tangents at the end of a focal chord PQ to a conic meet in T. The normal at P meets the axis in G and TQ in L. Show that PL varies directly as PG.*

H. G. Mayo.

Solution by C. E. Youngman.

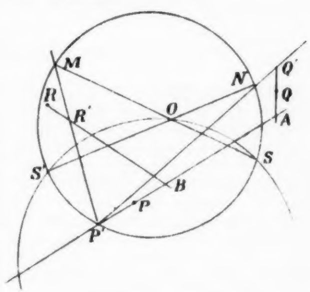
Let S be the focus not on PQ. In the triangle SPQ, PG and QT are bisectors of angles, one or both external; $\therefore$ they meet at L an excentre; opposite to P when the conic is an ellipse. Hence the projection of PL on $PQ=\frac{1}{2}(SP+PQ+QS)=AA'$ the focal axis. But the projection of PG on PS=semilatus-rectum; $\therefore$ $PG:PL=1-e^2:2$. Similarly for the hyperbola.

332. [R. 1. d.] *If P, P' be two successive positions of a point moving uniformly in a straight line, and Q, Q' two corresponding positions of a second point moving in like manner in another straight line, prove the following construction for the straight line which joins the moving points at the instant when they are collinear with a fixed point R. Draw the complete quadrilateral whose sides are PP', QQ', PQ, $P'Q'$, and let S be the point of intersection of the circles about its triangles, and let L be the pedal line of one of these triangles for the point S. On RS as diameter draw a circle cutting L in M and N. Each of the lines RM, RN is a solution of the problem.*

Modify the above construction for the case in which R is not a fixed point, but one moving uniformly in a straight line from R to R' while the first point moves from P to P'. R. F. MUIRHEAD.

Solution by C. E. YOUNGMAN.

Since $PP' : QQ' =$ const., the envelope of $P'Q'$ is a parabola (touching PP' and QQ'), of which the construction finds S the focus and L the vertical tangent, and by means of these the required tangents from R.



One may avoid mention of the parabola by regarding PP' and QQ', which meet at A, as parts of two figures directly similar. Then S their double point, found by the circles APQ, $AP'Q'$, is fixed, and its projection l on $P'Q'$, being collinear with those on AP' and AQ', describes a fixed line L. But when $RP'Q'$ is straight, l lies also on the semi-circle RS; $\therefore$ etc.

When R moves, let PP' and RR' meet at B. Find S' by the circles BPR, $BP'R'$; and draw the circle $P'SS'$ cutting $P'R'$ at M and $P'Q'$ at N. Then angle $S'SM = S'P'R' = S'BR'$ $=$ const.; $\therefore$ M moves on a fixed line SM. Similarly N on another $S'N$, meeting SM at O. When M is at O, so is N, and therefore both $P'R'$ and $P'Q'$ pass through O; *i.e.* $P'Q'R'$ are collinear. Thus the required lines join O to the points where PP' cuts the circle OSS'.

333. [K. 2. b.] *Through the vertex of any triangle can be drawn a circle touching the circumcircle, the incircle, and the excircle opposite that vertex; also a second circle touching the circumcircle and the remaining excircles; and the sum of the diameters of these two circles equals that of the circumcircle.*

C. E. M'VICKER.

Solution by C. E. YOUNGMAN.

From the vertex A invert, with $k^2 = s(s-a) = bc\cos^2\frac{1}{2}A$, the circles (I), (I_a) into each other and the circumcircle (O) into a straight line perpendicular to AO and therefore parallel to the fourth common tangent of (I), (I_a). This tangent then is the inverse of a circle passing through A and touching (O), (I), and (I_a). If d be its diameter, we have $dp = k^2$, where p is the distance of A from the fourth tangent, or from BC, and is given by $2Rp = bc$. Hence $d = 2R\cos^2\frac{1}{2}A$. In like manner, for the circle touching (O), (I_b), (I_c),

$$d' = 2R\sin^2\tfrac{1}{2}A\ ;\ \therefore\ d + d' = 2R.$$

Solutions have also been received from J. BLAIKIE and W. S. COONEY.

335. [I. 1.] *The minute hand of a clock in descending slips down at a constant rate of 1 minute space per 30 true minutes, and in ascending slips back at the same rate. It is set right at noon. Find the next two and the last occasions*

when it shows true time, and prove that in $37\frac{1}{2}$ *days it is* 1 *hr. late for the first time.* T. P. THOMPSON.

Solution by J. BLAIKIE and others.

The first 30 clock minutes $=30\times\dfrac{30}{31}$ or $29\frac{1}{31}$ true minutes, and since during the next half-hour the clock loses as much per real minute as it gained in the first half hour, it will be right at $2\times 29\frac{1}{31}$ minutes or 12 h. $58\frac{2}{31}$ m. It has now $1\frac{29}{31}$ clock minutes $=2\frac{2}{899}$ real minutes until the first clock hour is completed, and it will therefore be again right at $2\frac{2}{899}$ minutes past one o'clock.

During one clock hour the clock goes $30\left(\dfrac{30}{31}+\dfrac{30}{29}\right)$ or $\dfrac{60\times 900}{899}$ real minutes, that is $\dfrac{900}{899}$ hours; $\therefore$ 899 clock hours are equal to 900 real hours, that is $37\frac{1}{2}$ days, and the clock will have lost one hour in that interval.

NOTICES.

Arithmetic, Theoretical and Practical. By J. S. MACKAY, M.A., LL.D. (London and Edinburgh: W. & R. Chambers, 1899. Pp. 472.)

Dr. Mackay's *Arithmetic* has many striking features, and displays much originality; and even when he least agrees with the innovations advocated, a teacher will find much that is suggestive, and not a little that is valuable. In days when the local examiner is king upon the earth it needs some courage to write slanderous things of our old friends, Troy Weight and Apothecaries', and to declare that "to ordinary people they are of no use whatever, and to school pupils they are worse than useless"; while we have only to turn the page to find the deserved death doom of the rod, pole, or perch,—"for the ordinary purposes of life these details are of little use, and, as far as school instruction is concerned, should have been abandoned long ago." In Long Measure Dr. Mackay uses only the inch, foot, yard, chain, furlong, mile, and in Square Measure inch, foot, yard, chain, acre, mile. In Compound Multiplication the splitting of the multiplier into units, tens, hundreds, etc., is discarded, and the whole operation performed in one step,—with doubtful advantage. Two novelties are introduced into Long Division, which, no doubt, have been often practised before, but have never, we think, appeared in a text-book. An arrangement of Long Division due to Sang is revived. Complementary Addition is used for Subtraction. Multiplication slopes from left to right. In Decimals the multiplier and divisor are made to consist of a number containing only one integral figure. In factorisation it is recommended that after testing for divisibility by 2, 3, 5, 7, 11 the pupil should find the H.C.F. of his number and 146969($=47\times 53\times 59$), 2022161, 2800733, 800153, etc. In Evolution "a notable simplification of Horner's Method" is introduced. The "simplification" results in very little of Horner being left.

But we welcome the book chiefly because Dr. Mackay has gone as far perhaps as the local examiner will allow him to go in the direction of the higher arithmetic. His pages are full of suggestive hints from Lucas and elsewhere, and his collection of problems is very stimulating and largely original. W. P. WORKMAN.

Mémoire sur un nouvelle méthode pour la résolution des équations numériques, par HENRI PINET: suivi d'un appendice donnant le détail des opérations, par ÉMILE KRAUSS. (1899. Paris: Libraire Nony. 4to, 47 pp.)

M. Pinet, a distinguished French legist, has written a memoir which will be interesting to many of our readers, although the application to the resolution of numerical equations is perhaps not of very great value. By a "minimal decomposition" of a number M. Pinet indicates the expression of the number as a product of m factors $a_1, a_2, \ldots a_m$; m is a given number, and no factor a is greater than any subsequent factor a; subject to this, each factor a has the greatest value consistent with the conditions. It appears, by examples, that the sum of the

factors and the difference between the extreme factors are generally less for a minimal decomposition than for any non-minimal decomposition of the same number, m being also the same. For instance,

$$360=3\times4\times5\times6\ (\text{min.})=3\times3\times5\times8=2\times2\times3\times30=\text{etc.},$$

where the sum (18) and difference (3) are less for the minimal decomposition than for the others. M. Pinet gives an example,

$$942480=12\times14\times15\times17\times22\ (\text{min.})=11\times15\times16\times17\times21\ (\text{non-min.}),$$

in which the sum and difference for the minimal decomposition are the same as for the particular non-minimal decomposition shown. The question is akin to the relations between the geometric and arithmetic means of m quantities, but is complicated by the discontinuity introduced when the elements are required to be integral. (Cf. Chrystal, *Algebra*, chap. xxiv., § 8.) Perhaps some of our readers will investigate the matter. T.

Elementary Trigonometry. By A. J. PRESSLAND, M.A., and CHARLES TWEEDIE, M.A. (Oliver & Boyd, 1899. Pp. 313, xxx.)

Plane Trigonometry. By DANIEL A. MURRAY, Ph.D. (Longmans, Green & Co., 1899; pp. 206.)

A Short Course of Elementary Plane Trigonometry. By C. PENDLEBURY, M.A. (Bell, 1900. Pp. 160. 2s. 6d.)

The principal features of Messrs. Pressland and Tweedie's Trigonometry are:—the constant use of projection, neat methods of computation, the attention paid to graphs, and the study of geometrical applications at an early stage. The authors are familiar with the best French works on Trigonometry; but they are catholic in their tastes, and have utilised for their purpose many interesting results from Lodge and Langley, as well as from Chasles, Rouché, and Comberousse. Above all, they know their *Chrystal*, and use him frequently and judiciously. We give as novelties in the text of an elementary work, and in the admirable collection of examples (§ 101), the treatment of the type,

$$a+b\cos\beta+c\cos(\beta+\gamma)+d\cos(\beta+\gamma+\delta)=0,$$

and Ex. 35, p. 298 (Summation of Series), "Find the area of every loop of the curves $r=\sin 2\theta,\ r^2=\sin 2\theta,\ r^2=\cos 3\theta$."

Referring to the list of Tables on p. 146, to Schomilch's Five Figure Tables might be added those of Mougin (1 fr. 25 c.). We note that the term "pericycle" is used, apparently without definition.

Mr. Murray's contribution has as its aim to "avoid the extremes of expansion and brevity." He is at great pains to dwell upon the explanation of principles, and has endeavoured "to develop independence of mind and the power of mental initiative." Perhaps the best test of his success will be a few months' use of the book in class, and from a careful perusal of the volume we are inclined to think that time may substantiate his claim. Copious historical notes are a feature of interest. The Trigonometry, like the course on Differential Equations, has an eye to the requirements of the engineer.

Mr. Pendlebury's *Short Course* needs no commendation to those who are familiar with his work as a teacher and as a writer of school-books. It covers the ground up to Solution of Triangles and the "remarkable" circles.

Annuaire pour l'an 1900. Publié par le Bureau des Longitudes. (Gauthier-Villars, 1900. Pp. 795. 1 fr. 50 c.)

Into this compact little annual is packed an enormous mass of astronomical, chemical, and physical statistics. It also contains articles by M. Cornu on electric generators, by M. Lippmann on the new gases of the atmosphere, and by M. Janssen *re* his experiments on M. Blanc. Three hundred pages are devoted to astronomy, 130 pages to magnetic maps and physical tables; there are also 36 pages of tables of interest and annuities and rates of mortality.

A Short Table of Integrals. By Prof. B. O. PIERCE. Revised edition. (Ginn & Ed. Arnold, 1899. Pp. 134. 4s. 6d.)

Contains about 900 integrals and auxiliary formulas, with additional tables of natural logarithms and trigonometrical functions, common logarithms of Gamma

functions; values of e^x, e^{-x}, sinh x, cosh x, log sinh x, log cosh x, gd x, etc. Well printed in bold, clear type.

Mathématiques et Mathématiciens. By A Rebière. 3rd edition. (Nony, 1898. Pp. 566.)

A mine of curious lore, apothegms, *dicta mathematica*, amusing and instructive anecdotes of mathematicians, and of interesting and humorous paradoxes and problems. We have set one of the latter in the last number (No. 349). It is attributed to M. Lemoine, and is said to be as yet unanswered. A book for the odd half-hour!

BOOKS, MAGAZINES, ETC., RECEIVED.

Eléments de la Théorie des Nombres. Par E. CAHEN. (Paris: Gauthier-Villars, 1900. Pp. viii., 404.)

Our Patent Laws. By JAMES KEITH, C.E. (Unwin, 1900. Pp. 20.)

A First Geometry Book. By J. G. HAMILTON and F. KETTLE. (Ed. Arnold, 1900. Pp. 91. 1s.)

The Annals of Mathematics. Second Series: Vol. I., No. 1. Edited by Messrs. ORMOND STONE, W. E. BYERLY, H. S. WHITE, W. F. OSGOOD, and MAXIME BÔCHER. Pp. 49. (Harvard Univ.)

The American Mathematical Monthly. Vol. VI., Dec., 1899, Vol. VII., Jan.—March, 1900. Edited by Prof. B. F. FINKEL, M.Sc., and J. M. COLAW, A.M. (Springfield, Mo., U.S.A.)

Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen. Math.-phys. Klasse, 1899. Heft 3. (Horstmann, Göttingen.)

Math.-naturwissenschaftliche Mitteilungen im Auftrag des math.-naturwiss. Vereins in Württemberg. Edited by Dr. BÖKLEN and Dr. WÖLFFING. April, 1899—April, 1900. (Stuttgart.)

Periodico di Matemica. Anno XV. Fasc. IV., V. Edited by Prof. LAZZERI. (Livorno.)

Suppl. al Per. di Mat. Anno III. Fasc. IV., V.

Il Pitagora. Anno VI. Jan.—May, 1900. Edited by Prof. FAZZARI. (Palermo.)

Journal des Mathématiques Elémentaires. Jan.—April, 1900. Edited by Prof. G. MARIAUD. (Delagrave, Paris.)

Indian Engineering. Edited by PAT. DOYLE, C.E. Vol. XXVII., No. 5.

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